

Maximal super Yang-Mills theories on curved background with off-shell supercharges

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ABSTRACT: We construct $d \leq 7$ dimensional maximally supersymmetric Yang-Mills theories on a class of curved backgrounds with off-shell supercharges. The off-shell supersymmetry is mainly a generalization of on-shell supersymmetry previously constructed by Blau. We present several examples of backgrounds and discuss the number of the preserved supersymmetries on these backgrounds. We also construct another maximally supersymmetric Yang-Mills theories on S^3 obtained by dimensional reduction along \mathbb{R} -direction of $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R} \times S^3$.

KEYWORDS: Supersymmetric gauge theory, Extended supersymmetry, Localization.

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1. Introduction

The maximally supersymmetric Yang-Mills theories (SYM) have prominent structures at quantum level and deep connections with superstring theory [1, 2]. One of most typical examples is the four-dimensional $\mathcal{N} = 4$ SYM. Since the beta function of this theory vanishes, the theory is therefore known as conformal field theory [3, 4, 5, 6, 7]. Furthermore, it has been also proven that the theory is UV-finite [7, 8, 9]. In spite of such simple UV structures, the supersymmetric transformations for the component fields seem much more complicated than less supersymmetric gauge theories. This is because the supersymmetry

closes only on-shell and is not realized linearly. Actually we do not know whether fully off-shell formalism of the maximal SYM exists or not and this has been the one of long-standing problems in theoretical physics. In ref. [10], the authors have constructed the light-cone gauge fixed action of the four dimensional $\mathcal{N} = 4$ SYM, where eight supersymmetries close off-shell in this formalism. While this is quite useful for proofs of UV finiteness [9, 11, 12], the action is non-local and does not have manifest Lorentz and gauge invariance. In 1993, Berkovits has constructed the local off-shell supersymmetric action with manifest Lorentz and gauge invariance by introducing seven auxiliary fields [13]. One can show that the maximal number of the off-shell supercharges is nine [13, 14].

Off-shell supersymmetry plays an important role in theoretical physics. One of most important examples is that an off-shell supercharge in a theory enables us to apply localization method [15] to the theory in principle, which often drastically simplifies its analysis. However, even if we construct the off-shell supersymmetry in the flat space and apply the localization procedure, the partition function or correlation functions often suffer from divergences coming from the infrared effect and flat directions. The former is regularized by putting the theory on compact spaces and the latter is regularized by introducing mass terms. Pestun has solved these problems simultaneously by constructing partial off-shell supersymmetry on S^4 and shown that the localization works well¹. In this case, Killing spinors are modified and the mass terms associated with the curvature are introduced in order to close the supersymmetry algebra. Nevertheless, off-shell maximal SYM in other dimensions on curved background has not been studied so much while there have been recently remarkable developments for less supersymmetric cases [37, 38, 39] and from the point of view of holography [40, 41].

If we succeed in such constructions, we can apply the localization method to the maximal SYM and compute exactly some BPS observables in principle. Since the $(p+1)$ -dimensional maximal SYM is low-energy effective theory of Dp-branes, it probably enables us to understand nontrivial problems in superstring theory via the gauge/gravity duality [1, 2]. Among them, one of the interesting class is the five-dimensional maximal SYM. It has been conjectured that the six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theory on circle compactification is described by the five-dimensional maximal SYM [42, 43]. The $(2, 0)$ -theory is expected to contain N^3 -degrees of freedom from the gravity side analysis [44]. However, intrinsic off-shell description of the $(2, 0)$ -theory has not been found yet and it is difficult to derive the N^3 -behavior from the fields theory side directly. The explicit off-shell Lagrangian description of SYM on S^5 [30] make it possible to derive the N^3 -behavior via localization [45, 46]. On the other hand, for higher dimensional cases, off-shell supersymmetric actions on most curved spaces have not been constructed yet.

In this paper, we construct $d \leq 7$ dimensional maximally supersymmetric Yang-Mills theories on a class of curved backgrounds with off-shell supercharges by using the Berkovits method [13]. In particular, we mainly extend the on-shell supersymmetric action formu-

¹Recently many supersymmetric field theories on some curved backgrounds with rigid SUSY have been also constructed in the context of localization method for three dimensions [16, 17, 18, 19, 20, 21, 22, 23, 24, 25], four dimensions [26, 27, 28, 29] and five dimensions [30, 31, 32, 33, 34]. See also [35, 36] for a twist of rigid supersymmetry and the equivariant localization on Seifert manifolds.

lated in [47] to be off-shell. The maximal number of independent off-shell supersymmetries depends on the background while the number is nine for the flat space.

This paper is organized as follows. In section 2, we review the Berkovits construction, which has originally constructed the ten-dimensional super-Yang-Mills theory on the flat space with maximally 9 off-shell supercharges. In section 3, we construct the action on curved backgrounds admitting a class of Killing spinors. In section 4, we provide some concrete examples. In section 5, we introduce another class of three-dimensional maximal SYM on S^3 and the BMN matrix model [48]. Section 6 is devoted to a summary and discussions.

2. Maximal SYM on flat space with off-shell supercharges

In this section, we review the Berkovits method [13], which has originally constructed ten-dimensional SYM on the flat space with maximally 9 off-shell supercharges.

2.1 Action and supersymmetric transformation

Let us start with the on-shell supersymmetric action of the ten-dimensional $\mathcal{N} = 1$ SYM on $\mathbb{R}^{9,1}$ or \mathbb{R}^{10} :

$$S_{\text{on-shell}} = \frac{1}{g_{YM}^2} \int d^{10}x \text{Tr} \left[\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} \Psi \Gamma^M D_M \Psi \right], \quad (2.1)$$

where the fields take value in the real Lie algebra² and Ψ is the sixteen components Majorana-Weyl spinor. The action is invariant under the supersymmetric transformation

$$\begin{aligned} \delta_{\epsilon}^{(\text{on-shell})} A_M &= \epsilon \Gamma_M \Psi, \\ \delta_{\epsilon}^{(\text{on-shell})} \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon, \end{aligned} \quad (2.2)$$

where ϵ is the “bosonic” constant Majorana-Weyl spinor. It is well-known that the algebra generated by the transformation closes only on-shell since the off-shell bosonic degrees of freedom differ from the fermionic ones.

Here we add $16 - (10 - 1) = 7$ auxiliary fields to the action³ [13] as

$$S_{10d} = \frac{1}{g_{YM}^2} \int d^{10}x \text{Tr} \left[\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} \Psi \Gamma^M D_M \Psi - \frac{1}{2} K_m K_m \right], \quad (2.3)$$

and modify the transformation as

$$\begin{aligned} \delta_{\epsilon}^{(10d)} A_M &= \epsilon \Gamma_M \Psi, \\ \delta_{\epsilon}^{(10d)} \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + K_m \nu_m, \end{aligned}$$

²Here we basically use the notation of ref. [15]. See appendix. A for our conventions on the Clifford algebra. For the $U(N)$ gauge group, we take the generators T^a as anti-hermitian and normalization as $\text{Tr}(T^a T^b) = -\delta_{ab}$. Note that we define the Boltzmann weight as e^{+S} .

³Here we take K_m to be imaginary.

$$\delta_\epsilon^{(10d)} K_m = -\nu_m \Gamma^M D_M \Psi, \quad (2.4)$$

where $m = 1 \cdots 7$. ν_m is also bosonic Majorana-Weyl spinor satisfying

$$\begin{aligned} \epsilon \Gamma^M \nu_m &= 0, \\ \frac{1}{2}(\epsilon \Gamma_M \epsilon) \tilde{\Gamma}_{\alpha\beta}^M &= \nu_\alpha^m \nu_\beta^m + \epsilon_\alpha \epsilon_\beta, \\ \nu_m \Gamma^M \nu_n &= \delta_{mn} \epsilon \Gamma^M \epsilon. \end{aligned} \quad (2.5)$$

Note that δ_ϵ is fermionic since ϵ and ν_m are bosonic. We can show that the algebra generated by the transformation closes off-shell.

Let us perform dimensional reduction to d -dimensions for a later convenience. Then, we obtain the action of the maximal SYM on $\mathbb{R}^{d-1,1}$ or \mathbb{R}^d in the Berkovits formalism as

$$S_{\text{flat}} = \frac{1}{g_{YM}^2} \int d^d x \text{Tr} \left[\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} \Psi \Gamma^M D_M \Psi - \frac{1}{2} K_m K_m \right]. \quad (2.6)$$

A_M denote gauge fields and adjoint scalars:

$$A_{M=\mu} = A_\mu, \quad A_{M=A} = \Phi_A, \quad (2.7)$$

where $\mu = 1 \cdots d, A = d+1 \cdots 9, 0$. The field strength is explicitly given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad F_{\mu A} = D_\mu \Phi_A, \quad F_{AB} = [\Phi_A, \Phi_B], \quad (2.8)$$

and the covariant derivative is

$$D_\mu \Psi = \partial_\mu \Psi + [A_\mu, \Psi], \quad D_A \Psi = [\Phi_A, \Psi]. \quad (2.9)$$

This action is invariant under the transformation

$$\begin{aligned} \delta_\epsilon^{(\text{flat})} A_M &= \epsilon \Gamma_M \Psi, \\ \delta_\epsilon^{(\text{flat})} \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + K_m \nu_m, \\ \delta_\epsilon^{(\text{flat})} K_m &= -\nu_m \Gamma^M D_M \Psi. \end{aligned} \quad (2.10)$$

If we integrate K_m out, we can realize the well-known on-shell supersymmetric action and transformation.

2.2 The number of the off-shell supercharges

In the Berkovits method, the supersymmetric transformation (2.10) is generated by the 8 spinors ϵ, ν_m constrained by the condition (2.5). In order to count the number of the off-shell supercharges in conventional SUSY, we should rewrite the transformation (2.10) in the language of the single spinor ϵ by solving the constraint (2.5). In [14], the author has constructed the explicit solutions, which have 8, 9 and 4 linearly independent spinor parameters ϵ . In this section we introduce these solutions.

Solution with eight off-shell supersymmetries

Here we impose the restriction to ϵ as

$$\Gamma_{09}\epsilon = \epsilon, \quad (2.11)$$

giving eight linearly independent supercharges. Then one can show that the spinors

$$\nu_m = \Gamma_{m8}\epsilon \quad (2.12)$$

solve the constraint (2.5). It is convenient to use the following representation of the Clifford algebra:

$$\begin{aligned} \Gamma^0 &= \begin{pmatrix} \mathbf{1}_{8 \times 8} & 0 \\ 0 & \mathbf{1}_{8 \times 8} \end{pmatrix}, & \Gamma^9 &= \begin{pmatrix} \mathbf{1}_{8 \times 8} & 0 \\ 0 & -\mathbf{1}_{8 \times 8} \end{pmatrix}, \\ \Gamma^8 &= \begin{pmatrix} 0 & \mathbf{1}_{8 \times 8} \\ \mathbf{1}_{8 \times 8} & 0 \end{pmatrix}, & \Gamma^m &= \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix}, \end{aligned} \quad (2.13)$$

where λ_m is the anti-symmetric matrix satisfying⁴

$$\lambda_m \lambda_n + \lambda_n \lambda_m = -2\delta_{mn}. \quad (2.14)$$

This representation naturally decomposes ten-dimensional spinors into eigenstates of Γ_{09} as $\chi = (\chi^{(+)}, \chi^{(-)})$ with $\Gamma_{09}\chi^{(\pm)} = \pm\chi^{(\pm)}$. Then, the solution (2.12) is rewritten as

$$\nu_m^{(+)} = \lambda_m \epsilon^{(+)}, \quad \nu_m^{(-)} = \epsilon^{(-)} = 0. \quad (2.15)$$

Solution with nine off-shell supersymmetries

There is also a similar solution:

$$\nu_m^{(+)} = n \lambda_m n \epsilon^{(+)}, \quad \nu_m^{(-)} = -\lambda_m \epsilon^{(-)} = 0, \quad n \epsilon^{(-)} = \epsilon^{(-)}, \quad (2.16)$$

where $n_{\alpha\beta} = \text{diag}(-1, -1, -1, -1, -1, -1, -1, +1)$. While the projection of $\epsilon^{(-)}$ onto the eigenspinor of n restricts $\epsilon^{(-)}$ to one degree of freedom, the projection does not restrict $\epsilon^{(+)}$. Thus, the number of the off-shell SUSY is nine.

Solution with four off-shell supersymmetries

If we take ϵ and ν_m as

$$\begin{aligned} \Gamma_{45}\epsilon &= \Gamma_{67}\epsilon = \Gamma_{89}\epsilon, \\ \nu_1 &= \Gamma_{68}\epsilon = -\Gamma_{79}\epsilon, & \nu_2 &= -\Gamma_{69}\epsilon = -\Gamma_{78}\epsilon, \\ \nu_3 &= \Gamma_{84}\epsilon = -\Gamma_{95}\epsilon, & \nu_4 &= -\Gamma_{85}\epsilon = -\Gamma_{94}\epsilon, \\ \nu_5 &= \Gamma_{46}\epsilon = -\Gamma_{57}\epsilon, & \nu_6 &= -\Gamma_{47}\epsilon = -\Gamma_{56}\epsilon, \\ \nu_7 &= -\Gamma_{45}\epsilon = -\Gamma_{67}\epsilon = -\Gamma_{89}\epsilon, \end{aligned} \quad (2.17)$$

then this also solve the constraint (2.5). Since two linearly independent conditions are imposed, the solution leaves us four off-shell supersymmetries.

⁴This condition is satisfied by octonions.

3. Maximal SYM on curved space with off-shell supercharges

In this section, we construct the d -dimensional maximal SYM on a class of curved spaces with off-shell SUSY by extending the Berkovits method [13].

3.1 Killing spinor

In this paper, we construct the maximal SYM, which is invariant under the SUSY generated by Killing spinors satisfying

$$\nabla_\mu \epsilon = \alpha \tilde{\Gamma}_\mu \Gamma \epsilon, \quad (3.1)$$

where Γ is the product of “internal” gamma matrices. We denote the number of the internal gamma matrices by \sharp . The on-shell supersymmetric action invariant under the SUSY has been constructed in [47]. In order to be consistent with the chirality condition, we take \sharp to be odd below. In the Riemannian case, the Killing spinor equation implies the so-called first integrability condition:

$$R_{\mu\nu} = -4\alpha^2(-1)^\sharp(\Gamma)^2(d-1)g_{\mu\nu}. \quad (3.2)$$

Note that this condition is not sufficient for the existence of the Killing spinors⁵.

Supersymmetry on curved space admitting a type of Killing spinor including (3.1) has been studied in ref. [40]. First, introducing the Dirac operator $D = \Gamma^\mu \nabla_\mu$, we can rewrite the Killing spinor equation as

$$\nabla_\mu \epsilon = \frac{1}{d} \Gamma_\mu D \epsilon. \quad (3.3)$$

Note that $D\epsilon$ satisfies the following relation from the Killing spinor eq. (3.1)

$$\nabla_\mu(D\epsilon) = d\lambda^2 \tilde{\Gamma}_\mu \epsilon \quad \text{with} \quad \lambda^2 = -\alpha^2(\Gamma)^2. \quad (3.4)$$

By using these relations, one can easily show that the spinors $\epsilon_\pm = \mp \lambda \epsilon - \frac{1}{d} D\epsilon$ satisfy

$$\nabla_\mu \epsilon_\pm = \pm \lambda \Gamma_\mu \epsilon_\pm. \quad (3.5)$$

The Riemannian manifolds admitting this type of Killing spinor equation have been classified well in refs. [49, 50, 51]. For real λ case, it has been shown to be admitted on non-compact manifolds, which are a warped product of \mathbb{R} with any $(d-1)$ -dimensional manifolds [49]. For purely imaginary λ case, the situation becomes somewhat complicated. The existence of the Killing spinor on the space \mathcal{M}_d implies the existence of a covariantly constant spinor on the cone $C(\mathcal{M}_d)$ [50]. For example, the four-dimensional space \mathcal{M}_4 should be conformally flat manifold. For $d=3$, it is admitted for $\mathcal{M}_3 = S^3$ or its quotients. In higher dimensions, the Killing spinor is admitted for Sasaki-Einstein manifolds in five dimensions and nearly-Kähler manifolds in six dimensions, respectively. In sec. 4, we provide some concrete examples.

⁵In the pseudo-Riemannian case, the condition (3.2) is neither necessary nor sufficient except for $d=1$. For details, see [47].

3.2 Construction of action

Since the Killing spinors on a curved space are no longer covariantly constants ($\nabla_\mu \epsilon \neq 0$) generically, the naive extension of the flat action (2.6)

$$S_0 = \frac{1}{g_{YM}^2} \int d^d x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} \Psi \Gamma^M D_M \Psi - \frac{1}{2} K_m K_m \right] \quad (3.6)$$

is not invariant under the transformation

$$\begin{aligned} \delta_\epsilon^{(0)} A_M &= \epsilon \Gamma_M \Psi, \\ \delta_\epsilon^{(0)} \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + K_m \nu_m, \\ \delta_\epsilon^{(0)} K_m &= -\nu_m \Gamma^M D_M \Psi. \end{aligned} \quad (3.7)$$

Actually the variation of the action under $\delta_\epsilon^{(0)}$ is

$$\begin{aligned} g_{YM}^2 \delta_\epsilon^{(0)} S_0 &= \int d^d x \sqrt{g} \text{Tr} \left[-\frac{1}{2} F_{NP} (\nabla_\mu \epsilon) \Gamma^{NP\mu} \Psi + F^{\mu N} (\nabla_\mu \epsilon) \Gamma_N \Psi \right] \\ &= (-1)^{\Gamma+1} \int d^d x \sqrt{g} \text{Tr} \left[\frac{\alpha(d-4)}{2} F_{\mu\nu} (\epsilon \Gamma \Gamma^{\mu\nu} \Psi) + \alpha(d-2) F_{\mu A} (\epsilon \Gamma \Gamma^{\mu A} \Psi) \right. \\ &\quad \left. + \frac{\alpha d}{2} F_{AB} (\epsilon \Gamma \Gamma^{AB} \Psi) \right], \end{aligned} \quad (3.8)$$

where $(-1)^\Gamma$ is defined by

$$\Gamma^T = (-1)^\Gamma \Gamma, \quad (3.9)$$

in terms of the transposition Γ^T . This cannot vanish except for the case $\nabla_\mu \epsilon = 0$ ($\alpha = 0$), namely, Ricci-flat case. Therefore in order to obtain desired actions, we must deform the above action and transformation. We consider the well-known case for $d = 4$ and then the less-known case for $1 \leq d \leq 7$. The properties of SUSY algebra will be investigated in sec. 3.3.

3.2.1 Class 1 ($d = 4$)

Let us consider the following action

$$S_1 = S_0 + \frac{1}{g_{YM}^2} \int d^d x \sqrt{g} \text{Tr} \left[\frac{c_\Phi}{2} \Phi_A \Phi^A \right], \quad (3.10)$$

and transformation

$$\begin{aligned} \delta_\epsilon^{(1)} A_M &= \epsilon \Gamma_M \Psi, \\ \delta_\epsilon^{(1)} \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + c \Phi_A \tilde{\Gamma}^A \Gamma \epsilon + K_m \nu_m, \\ \delta_\epsilon^{(1)} K_m &= -\nu_m \Gamma^M D_M \Psi. \end{aligned} \quad (3.11)$$

The transformation under $\delta_\epsilon^{(1)}$ is

$$\begin{aligned}
& g_{YM}^2 \delta_\epsilon^{(1)} S_1 \\
&= (-1)^{\Gamma+1} \int d^3x \sqrt{g} \text{Tr} \left[\frac{\alpha(d-4)}{2} F_{\mu\nu} (\epsilon \Gamma^{\mu\nu} \Psi) + (\alpha(d-2) + c) F_{\mu A} (\epsilon \Gamma^{\mu A} \Psi) \right. \\
&\quad \left. + \left(\frac{\alpha d}{2} + c \right) F_{AB} (\epsilon \Gamma^{AB} \Psi) + (-1)^{\Gamma+1} (c_\Phi + c\alpha d(\Gamma)^2) \Phi_A (\epsilon \Gamma^A \Psi) \right]. \tag{3.12}
\end{aligned}$$

The condition for invariance is

$$\alpha(d-4) = 0, \quad \alpha(d-2) + c = 0, \quad \frac{\alpha d}{2} + c = 0, \quad c_\Phi + c\alpha d(\Gamma)^2 = 0. \tag{3.13}$$

This condition has a nontrivial solution only for $d = 4$:

$$c = -2\alpha, \quad c_\Phi = 8\alpha^2(\Gamma)^2. \tag{3.14}$$

Thus, the action

$$\begin{aligned}
S_{4d} = \frac{1}{g_{YM}^2} \int d^4x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{MN} F^{MN} + 4\alpha^2(\Gamma)^2 \Phi_A \Phi^A \right. \\
\left. - \frac{1}{2} \Psi \Gamma^M D_M \Psi - \frac{1}{2} K_m K_m \right], \tag{3.15}
\end{aligned}$$

is invariant under the transformation

$$\begin{aligned}
\delta_\epsilon^{(4d)} A_M &= \epsilon \Gamma_M \Psi, \\
\delta_\epsilon^{(4d)} \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon - 2\alpha \Phi_A \tilde{\Gamma}^A \Gamma \epsilon + K_m \nu_m, \\
\delta_\epsilon^{(4d)} K_m &= -\nu_m \Gamma^M D_M \Psi. \tag{3.16}
\end{aligned}$$

Remarks

Since $4\alpha^2(\Gamma)^2 = \frac{R}{12}$, we have just recovered the so-called conformal mass term for the scalars. Actually we can obtain the action (3.15) by performing Weyl transformation from the flat action (2.6) for conformally flat spaces [15, 28]. The authors have considered more general (conformal) Killing spinors satisfying

$$\nabla_\mu \epsilon = \tilde{\Gamma}_\mu \tilde{\epsilon}, \tag{3.17}$$

which is nothing but the conformal Killing spinor equation. In this construction, the numbers of the on-shell and off-shell supercharges are thirty-two and eighteen, respectively.

3.2.2 Class 2 ($d \leq 7, \Gamma = \Gamma^{789}$)

Let us take⁶ $d \leq 7$ and $\Gamma = \Gamma^{789}$. Let us consider the action

$$S_2 = S_0 + \frac{1}{g_{YM}^2} \int d^d x \sqrt{g} \text{Tr} \left[\frac{c_\Phi}{2} \Phi_A \Phi^A + \frac{c'_\Phi}{2} \Phi_p \Phi^p + c_Y \epsilon_{pqr} \Phi^p [\Phi^q, \Phi^r] + \frac{c_\Psi}{2} \Psi \Gamma^{789} \Psi \right],$$

⁶Of course we can arbitrarily take other combinations of three indices from the internal space.

$$(3.18)$$

and the transformation

$$\begin{aligned}\delta_\epsilon^{(2)} A_M &= \epsilon \Gamma_M \Psi, \\ \delta_\epsilon^{(2)} \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + c \Phi_A \tilde{\Gamma}^A \Gamma^{789} \epsilon + c' \Phi_p \tilde{\Gamma}^p \Gamma^{789} \epsilon + K_m \nu_m, \\ \delta_\epsilon^{(2)} K_m &= -\nu_m \Gamma^M D_M \Psi + c_K \nu_m \Gamma^{789} \Psi,\end{aligned}\tag{3.19}$$

where $p, q, r = 7, 8, 9$ and $\hat{A} = d+1, d+2, \dots, 6, 0$. The variation under $\delta_\epsilon^{(2)}$ is

$$\begin{aligned}& g_{YM}^2 \delta_\epsilon^{(2)} S_2 \\ &= \int d^d x \sqrt{g} \text{Tr} \left[\frac{\alpha(d-4) - c_\Psi}{2} F_{\mu\nu} (\epsilon \Gamma^{789} \Gamma^{\mu\nu} \Psi) \right. \\ &\quad + (\alpha(d-2) + c + c_\Psi + c') F_{\mu p} (\epsilon \Gamma^{789} \Gamma^{\mu p} \Psi) + (\alpha(d-2) + c - c_\Psi) F_{\mu \hat{A}} (\epsilon \Gamma^{789} \Gamma^{\mu \hat{A}} \Psi) \\ &\quad + \left(\frac{\alpha d - c_\Psi}{2} + c - 3c_Y + c' \right) F_{pq} (\epsilon \Gamma^{789} \Gamma^{pq} \Psi) + (\alpha d + c_\Psi + 2c + c') F_{p \hat{A}} (\epsilon \Gamma^{789} \Gamma^{p \hat{A}} \Psi) \\ &\quad + \left(\frac{\alpha d - c_\Psi}{2} + c \right) F_{\hat{A} \hat{B}} (\epsilon \Gamma^{789} \Gamma^{\hat{A} \hat{B}} \Psi) \\ &\quad + (c_\Phi - c\alpha d + c_\Psi c - c' \alpha d + c_\Psi c' + c'_\Phi) \Phi_p (\epsilon \Gamma^p \Psi) + (c_\Phi - c\alpha d - c_\Psi c) \Phi_{\hat{A}} (\epsilon \Gamma^{\hat{A}} \Psi) \\ &\quad \left. + (c_K - c_\Psi) K_m \Psi \Gamma^{789} \nu_m \right].\end{aligned}\tag{3.20}$$

Therefore the condition for invariance is

$$\begin{aligned}c_\Psi &= \alpha(d-4), \quad \alpha(d-2) + c + c_\Psi + c' = 0, \quad \alpha(d-2) + c - c_\Psi = 0, \\ \alpha d - c_\Psi + 2c - 6c_Y + 2c' &= 0, \quad \alpha d + c_\Psi + 2c + c' = 0, \quad \alpha d - c_\Psi + 2c = 0, \\ c_\Phi - c\alpha d + c_\Psi c - c' \alpha d + c_\Psi c' + c'_\Phi &= 0, \quad c_\Phi - c\alpha d - c_\Psi c = 0, \quad c_K = c_\Psi.\end{aligned}\tag{3.21}$$

This has the following unique nontrivial solution:

$$\begin{aligned}c_\Phi &= -4\alpha^2(d-2), \quad c'_\Phi = -4\alpha^2(d-4), \quad c_\Psi = \alpha(d-4), \quad c_Y = -\frac{2\alpha(d-4)}{3}, \\ c &= -2\alpha, \quad c' = -2\alpha(d-4), \quad c_K = c_\Psi = \alpha(d-4).\end{aligned}\tag{3.22}$$

Thus, the action

$$\begin{aligned}S_2 &= \frac{1}{g_{YM}^2} \int d^d x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{MN} F^{MN} - 2\alpha^2(d-2) \Phi_A \Phi^A - 2\alpha^2(d-4) \Phi_p \Phi^p \right. \\ &\quad \left. - \frac{2\alpha(d-4)}{3} \epsilon_{pqr} \Phi^p [\Phi^q, \Phi^r] - \frac{1}{2} \Psi \Gamma^M D_M \Psi + \frac{\alpha(d-4)}{2} \Psi \Gamma^{789} \Psi - \frac{1}{2} K_m K_m \right],\end{aligned}\tag{3.23}$$

is invariant under the transformation

$$\delta_\epsilon^{(2)} A_M = \epsilon \Gamma_M \Psi,$$

$$\begin{aligned}
\delta_\epsilon^{(2)}\Psi &= \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon - 2\alpha\Phi_A\tilde{\Gamma}^A\Gamma^{789}\epsilon - 2\alpha(d-4)\Phi_p\tilde{\Gamma}^p\Gamma^{789}\epsilon + K_m\nu_m, \\
\delta_\epsilon^{(2)}K_m &= -\nu_m\Gamma^M D_M\Psi + \alpha(d-4)\nu_m\Gamma^{789}\Psi.
\end{aligned} \tag{3.24}$$

If we set $d = 4$, $\Gamma^{789} \rightarrow \Gamma$, this is same as the one of class 1.

Remarks

Since the action is complex except for $d = 4$, the theory seems to be not reflection positive. Similar situations have appeared in less supersymmetric gauge theories [38]. Therefore we consider that our result is their highly supersymmetric version.

For the $d = 2$ and purely imaginary α case, the scalars Φ_7, Φ_8 and Φ_9 have negative square of mass in spite of the positive curvature of the background. We are suspicious to that the theory for the case is not well-defined.

3.3 Supersymmetry algebra

In this section, we investigate properties of the SUSY algebras generated by the transformations (3.16) and (3.24) and confirm that these algebras close off-shell.

3.3.1 Class 1 ($d = 4$)

Let us consider the square of the supersymmetric transformation for each field. According to ref. [15], we obtain

$$\begin{aligned}
(\delta_\epsilon^{(4d)})^2 A_\mu &= -v^\nu F_{\nu\mu} - [v^A \Phi_A, D_\mu], \\
(\delta_\epsilon^{(4d)})^2 \Phi_A &= -v^\mu D_\mu \Phi_A - [v^B \Phi_B, \Phi_A] - \bar{R}_{AB} \Phi^B - \Omega \Phi_A, \\
(\delta_\epsilon^{(4d)})^2 \Psi &= -v^\mu D_\mu \Psi - [v^A \Phi_A, \Psi] - \frac{1}{4}(-\bar{R}_{\mu\nu} \Gamma^{\mu\nu} + \bar{R}_{AB} \Gamma^{AB}) \Psi - \frac{3}{2} \Omega \Psi, \\
(\delta_\epsilon^{(4d)})^2 K_m &= -v^\mu D_\mu K_m - [v^A \Phi_A, K_m] - (\nu_{[m} \not{D} \nu_{n]}) K_n - 2\Omega K_m,
\end{aligned} \tag{3.25}$$

where

$$v^M = \epsilon \Gamma^M \epsilon, \quad \bar{R}_{MN} = 2\alpha(\epsilon \tilde{\Gamma}_{MN} \Gamma \epsilon), \quad \Omega = 2\alpha(\epsilon \Gamma \epsilon). \tag{3.26}$$

Therefore the square of the supersymmetry transformation can be written as

$$(\delta_\epsilon^{(4d)})^2 = -L_v - G_\Phi - \bar{R} - \Omega. \tag{3.27}$$

Here L_v , G_Φ and \bar{R} are a Lie derivative in the direction of the (conformal Killing) vector field v^μ , the gauge transformation generated by the adjoint valued scalar field $\Phi = v^A \Phi_A$ and a transformation of coordinate or R -symmetry transformation, respectively. Finally Ω generates a local dilatation with the parameter $2\alpha(\epsilon \Gamma \epsilon)$. Note that the dilatation Ω vanishes for $\Gamma = \Gamma^{789}$ corresponding to the case for $d = 4$ of the class 2.

3.3.2 Class 2 ($d \leq 7, \Gamma = \Gamma^{789}$)

After some tedious calculations, we can obtain

$$\begin{aligned}
(\delta_\epsilon^{(2)})^2 A_\mu &= -v^\nu F_{\nu\mu} - [v^A \Phi_A, D_\mu], \\
(\delta_\epsilon^{(2)})^2 \Phi_{\hat{A}} &= -v^\nu D_\nu \Phi_{\hat{A}} - [v^B \Phi_B, \Phi_{\hat{A}}] - \bar{R}_{\hat{A}\hat{B}} \Phi^{\hat{B}}, \\
(\delta_\epsilon^{(2)})^2 \Phi_p &= -v^\nu D_\nu \Phi_p - [v^B \Phi_B, \Phi_p] - (d-3) \bar{R}_{pq} \Phi^q, \\
(\delta_\epsilon^{(2)})^2 \Psi &= -v^\nu D_\nu \Psi - [v^A \Phi_A, \Psi] - \frac{1}{4} (-\bar{R}_{\mu\nu} \Gamma^{\mu\nu} + \bar{R}_{\hat{A}\hat{B}} \Gamma^{\hat{A}\hat{B}} + (d-3) \bar{R}_{pq} \Gamma^{pq}) \Psi, \\
(\delta_\epsilon^{(2)})^2 K_m &= -v^\mu D_\mu K_m - [v^A \Phi_A, K_m] - (\nu_{[m} \not{D} \nu_{n]}) K_n,
\end{aligned} \tag{3.28}$$

where

$$\bar{R}_{MN} = 2\alpha(\epsilon \tilde{\Gamma}_{MN} \Gamma^{789} \epsilon). \tag{3.29}$$

Therefore the square of the supersymmetry transformation can be written as

$$(\delta_\epsilon^{(2)})^2 = -L_v - G_\Phi - \bar{R}. \tag{3.30}$$

Note that the dilatation vanishes automatically in this class as mentioned above. Also for the $d = 3$ case, R-symmetry transformation of the internal space $p = 7, 8, 9$ vanishes.

4. Some examples

In this section, we provide some concrete examples for the class 2. As we remarked in sec. 3.2.1, the class 1 is applicable for conformally flat manifolds. If we impose the conformal Killing spinor equation (3.17) as in [15, 28], the numbers of the on-shell and off-shell supercharges are thirty-two and eighteen, respectively.

4.1 S^d ($d \leq 7$)

For the class 2, the Killing spinor equation becomes

$$\nabla_\mu \epsilon = \alpha \tilde{\Gamma}_\mu \Gamma^{789} \epsilon. \tag{4.1}$$

If we rewrite the equation as (3.5) and take $\alpha = \frac{i}{2l}$ or $\alpha = -\frac{i}{2l}$,

$$\nabla_\mu \epsilon = \frac{i}{2l} \Gamma_\mu \epsilon \quad \text{or} \quad \nabla_\mu \epsilon = -\frac{i}{2l} \Gamma_\mu \epsilon, \tag{4.2}$$

which are same as the Killing spinor equations on S^d with radius l [52]. Actually, from the first integrability condition (3.2), the curvature is given by

$$R = \frac{d(d-1)}{l^2}, \tag{4.3}$$

which is nothing but the curvature of S^d . Since the equation (4.2) does not reduce the degrees of freedom of ϵ , the number of “on-shell” SUSY is sixteen. If we consider the solution (2.16) same as for the flat space one can easily show that the number of “off-shell” SUSY is nine.

4.2 AdS_d ($d \leq 7$)

If we set $\alpha = \frac{1}{2l}$ or $\alpha = -\frac{1}{2l}$, the Killing spinor equation becomes same as the one on AdS_d with radius l [52]. From the first integrability condition (3.2), the curvature is given by

$$R = -\frac{d(d-1)}{l^2}, \quad (4.4)$$

which is the curvature of AdS_d . Similarly for the S^d case, the numbers of the on-shell and off-shell supercharges are sixteen and nine, respectively.

4.3 S^3/\mathbb{Z}_n

The lens space S^3/\mathbb{Z}_n is obtained by \mathbb{Z}_n identification of S^3 , which is contained in the subgroup of the isometry group $U(1)_R \subset SU(2)_R \subset SU(2)_L \times SU(2)_R$ [24]. Since the Killing spinors on S^3 are constants along $SU(2)_L$ or $SU(2)_R$, the latter is also the Killing spinor on S^3/\mathbb{Z}_n . Thus, the numbers of the on-shell and off-shell supercharges are eight and at least four, respectively.

5. Another class of 3d $\mathcal{N} = 8$ SYM on S^3 and the BMN matrix model

In this section, we construct another class of maximal SYM on S^3 and the BMN matrix model [48] with off-shell SUSY. These theories are obtained by dimensional reducing 4d $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$.

5.1 Another class of 3d $\mathcal{N} = 8$ SYM on S^3

Let us start with 4d $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. It is well known that $\mathbb{R} \times S^3$ is the conformally flat manifold since

$$\begin{aligned} ds_{\mathbb{R}^4}^2 &= dr^2 + r^2 d\Omega_3^2 \\ &= e^{\frac{2}{l}\tau} (d\tau^2 + l^2 d\Omega_3^2) \quad (r = le^{\frac{\tau}{l}}) \\ &= e^{\frac{2}{l}\tau} ds_{\mathbb{R} \times S^3}^2. \end{aligned} \quad (5.1)$$

This means that the theory on $\mathbb{R} \times S^3$ is obtained by the Weyl transformation $g_{\mu\nu}^{\mathbb{R} \times S^3} = e^{2\omega} g_{\mu\nu}^{\mathbb{R}^4}$ with $\omega = -\frac{\tau}{l}$ from the theory on \mathbb{R}^4 . Since the curvature of $\mathbb{R} \times S^3$ is $\frac{6}{l^2}$, we obtain

$$\begin{aligned} S_{\mathbb{R} \times S^3} &= \frac{1}{g_{YM}^2} \int d\tau d\Omega_3 \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \Phi_A D^\mu \Phi^A + \frac{1}{2l^2} \Phi_A \Phi^A \right. \\ &\quad \left. + \frac{1}{4} [\Phi_A, \Phi_B][\Phi^A, \Phi^B] - \frac{1}{2} \Psi \Gamma^\mu D_\mu \Psi - \frac{1}{2} \Psi \Gamma^A [\Phi_A, \Psi] - \frac{1}{2} K_m K_m \right], \end{aligned} \quad (5.2)$$

which is invariant under the transformation

$$\begin{aligned} \delta_\epsilon^{(\mathbb{R} \times S^3)} A_M &= \epsilon \Gamma_M \Psi, \\ \delta_\epsilon^{(\mathbb{R} \times S^3)} \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + \frac{1}{2} \Gamma^{\mu A} \Phi_A \nabla_\mu \epsilon + K_m \nu_m, \end{aligned}$$

$$\delta_\epsilon^{(\mathbb{R} \times S^3)} K_m = -\nu_m \Gamma^M D_M \Psi. \quad (5.3)$$

The conformal Killing spinor satisfies

$$\nabla_\mu \epsilon = \tilde{\Gamma}_\mu \tilde{\epsilon}, \quad \nabla_\mu \tilde{\epsilon} = -\frac{1}{8l^2} \Gamma_\mu \epsilon, \quad (5.4)$$

which are solved as

$$\epsilon = e^{\frac{1}{2}\omega}(\epsilon_s + x^\mu \tilde{\Gamma}_\mu \epsilon_c), \quad \tilde{\epsilon} = e^{\frac{1}{2}\omega} \left(\frac{1}{2} e^{-\omega} \epsilon_c - \frac{1}{2l^2} e^\omega \Gamma^a x_a \epsilon_s \right), \quad (5.5)$$

where ϵ_s and ϵ_c are constant spinors. The supersymmetry algebra generated by the spinors is [15]

$$\begin{aligned} (\delta_\epsilon^{(\mathbb{R} \times S^3)})^2 A_\mu &= -v^\nu F_{\nu\mu} - [v^A \Phi_A, D_\mu], \\ (\delta_\epsilon^{(\mathbb{R} \times S^3)})^2 \Phi_A &= -v^\mu D_\mu \Phi_{\hat{A}} - [v^B \Phi_B, \Phi_A] - \bar{R}_{AB} \Phi^B - \Omega \Phi_A, \\ (\delta_\epsilon^{(\mathbb{R} \times S^3)})^2 \Psi &= -v^\mu D_\mu \Psi - [v^A \Phi_A, \Psi] - \frac{1}{4} (-\bar{R}_{\mu\nu} \Gamma^{\mu\nu} + \bar{R}_{AB} \Gamma^{AB}) \Psi - \frac{3}{2} \Omega \Psi, \\ (\delta_\epsilon^{(\mathbb{R} \times S^3)})^2 K_m &= -v^\mu D_\mu K_m - [v^A \Phi_A, K_m] - (\nu_{[m} \not{D} \nu_{n]}) K_n - 2\Omega K_m, \end{aligned} \quad (5.6)$$

where

$$v^M = \epsilon \Gamma^M \epsilon, \quad \bar{R}_{MN} = 2(\epsilon \tilde{\Gamma}_{MN} \tilde{\epsilon}), \quad \Omega = 2(\epsilon \tilde{\epsilon}). \quad (5.7)$$

Since the four-dimensional $\mathcal{N} = 4$ SYM is the conformal field theory, the dilatation Ω is the one of the symmetry and therefore harmless. However, the nonzero dilatation becomes harmful after the dimensional reduction. Thus, we should impose the constraint $\epsilon \tilde{\epsilon} = 0$, which reduces the degrees of freedom of ϵ to sixteen.

In order to derive the action of 3d $\mathcal{N} = 8$ SYM on S^3 , we apply dimensional reduction along \mathbb{R} -direction. Expanding the gauge field as $A = \phi d\tau + A^i e^i$, we derive the field strength as

$$\begin{aligned} F &= dA + A \wedge A \\ &= \left(-i \frac{2}{l} (J_i \phi) + (\partial_\tau A_i) + [\phi, A_i] \right) d\tau \wedge e^i \\ &\quad + \frac{1}{2} \epsilon_{ijk} \left\{ \frac{2}{l} i \epsilon_{klm} J_l A_m + \frac{2}{l} A_k + \frac{1}{2} \epsilon_{klm} [A_l, A_m] \right\} e^i \wedge e^j, \end{aligned} \quad (5.8)$$

where e^i ($i = 1, 2, 3$) is the left-invariant 1-form on S^3 and J_i is the Killing vector. The covariant derivative of fermion is rewritten as

$$\Gamma^\mu D_\mu \Psi = \Gamma^\tau D_\tau \Psi + \frac{2i}{l} \Gamma^i J_i \Psi + \frac{3i}{4l} \Gamma^{123} \Psi + \Gamma^i [A_i, \Psi]. \quad (5.9)$$

Applying dimensional reduction along the τ -direction, we obtain the following action different from the class 2:

$$S_3 = \frac{1}{g_{YM}^2} \int d\Omega_3 \text{Tr} \left[\frac{1}{2} (\tilde{D}_i \phi)^2 + \frac{1}{2} f_{ij} f^{ij} + \frac{1}{2} \tilde{D}_i \Phi_A \tilde{D}^i \Phi^A + \frac{1}{2l^2} \Phi_A \Phi^A + \frac{1}{2} [\phi, \Phi_A] [\phi, \Phi^A] \right]$$

$$+\frac{1}{4}[\Phi_A, \Phi_B][\Phi^A, \Phi^B] - \frac{1}{2}\Psi\Gamma^i\tilde{D}_i\Psi - \frac{3i}{8l}\Psi\Gamma^{123}\Psi - \frac{1}{2}\Psi\Gamma^\tau[\phi, \Psi] - \frac{1}{2}\Psi\Gamma^A[\Phi_A, \Psi] - \frac{1}{2}K_m K_m \Big], \quad (5.10)$$

where

$$\tilde{D}_i\phi = i\frac{2}{l}J_i\phi + [A_i, \phi], \quad f_{ij} = \frac{1}{2}\epsilon_{ijk} \left(\frac{2}{l}i\epsilon_{klm}J_l A_m + \frac{2}{l}A_k + \frac{1}{2}\epsilon_{klm}[A_l, A_m] \right). \quad (5.11)$$

This action is invariant under the transformation

$$\begin{aligned} \delta_\epsilon^{(3)}A_i &= \epsilon\Gamma_i\Psi, \quad \delta_\epsilon^{(3)}\phi = \epsilon\Gamma_\tau\Psi, \quad \delta_\epsilon^{(3)}\Phi_A = \epsilon\Gamma_A\Psi, \\ \delta_\epsilon^{(3)}\Psi &= \tilde{D}_i\phi\Gamma^{i\tau}\epsilon + \frac{1}{2}f_{ij}\Gamma^{ij}\epsilon + \tilde{D}_i\Phi_A\Gamma^{iA}\epsilon + [\phi, \Phi_A]\Gamma^{\tau A}\epsilon + \frac{1}{2}[\Phi_A, \Phi_B]\Gamma^{AB}\epsilon - 2\Phi_A\tilde{\Gamma}^A\tilde{\epsilon} + K_m\nu_m, \\ \delta_\epsilon^{(3)}K_m &= -\nu_m\Gamma^i\tilde{D}_i\Psi - \frac{3i}{4l}\nu_m\Gamma^{123}\Psi - \nu_m\Gamma^\tau[\phi, \Psi] - \nu_m\Gamma^A[\Phi_A, \Psi]. \end{aligned} \quad (5.12)$$

One of the main differences from the class 2 is the R-symmetry: $SO(6)_R$ in this class, while $SO(4)_R \times SO(3)_R$ in the class 2. Also, note that this action is real and therefore reflection positive different from the class2. The numbers of the on-shell and off-shell supercharges are sixteen and nine, respectively.

Similarly, we can generate three different maximal SYM's on S^2 . One is directly generated by the class 2 construction. The others are constructed by the dimensional reduction along the S^1 -fibre direction of the two different SYM's on S^3 .

5.2 The BMN matrix model

The action of the BMN matrix model [48] is obtained by dimensional reduction of the $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ along the S^3 -direction [53]. In the same way as (5.8)-(5.12) expanding the gauge field as $A = A_\tau d\tau + X^i e^i$, we obtain

$$\begin{aligned} S_{\text{BMN}} &= \frac{1}{g_{\text{YM}}^2} \int d\tau \text{Tr} \left[\frac{1}{2}D_\tau X_i D^\tau X^i + \frac{1}{2} \left(\frac{2}{l}X_i + \frac{1}{2}\epsilon_{ijk}[X_j, X_k] \right)^2 \right. \\ &\quad + \frac{1}{2}D_\tau \Phi_A D^\tau \Phi^A + \frac{1}{2l^2}\Phi_A \Phi^A + \frac{1}{2}[X_i, \Phi_A][X^i, \Phi^A] + \frac{1}{4}[\Phi_A, \Phi_B][\Phi^A, \Phi^B] \\ &\quad \left. - \frac{1}{2}\Psi\Gamma^\tau D_\tau \Psi - \frac{3i}{8l}\Psi\Gamma^{123}\Psi - \frac{1}{2}\Psi\Gamma^i[X_i, \Psi] - \frac{1}{2}\Psi\Gamma^A[\Phi_A, \Psi] - \frac{1}{2}K_m K_m \right] \end{aligned} \quad (5.13)$$

This action is invariant under the transformation

$$\begin{aligned} \delta_\epsilon^{(\text{BMN})}A_\tau &= \epsilon\Gamma_\tau\Psi, \quad \delta_\epsilon^{(\text{BMN})}X_i = \epsilon\Gamma_i\Psi, \quad \delta_\epsilon^{(\text{BMN})}\Phi_A = \epsilon\Gamma_A\Psi, \\ \delta_\epsilon^{(\text{BMN})}\Psi &= D_\tau X_i \Gamma^{\tau i}\epsilon + \frac{1}{2}f_{ij}\Gamma^{ij}\epsilon + D_\tau \Phi_A \Gamma^{\tau A}\epsilon \\ &\quad + [X_i, \Phi_A]\Gamma^{iA}\epsilon + \frac{1}{2}[\Phi_A, \Phi_B]\Gamma^{AB}\epsilon - 2\Phi_A\tilde{\Gamma}^A\tilde{\epsilon} + K_m\nu_m, \\ \delta_\epsilon^{(\text{BMN})}K_m &= -\nu_m\Gamma^\tau D_\tau \Psi - \frac{3i}{4l}\nu_m\Gamma^{123}\Psi - \nu_m\Gamma^i\tau[X_i, \Psi] - \nu_m\Gamma^A[\Phi_A, \Psi]. \end{aligned} \quad (5.14)$$

6. Conclusion

We have constructed the $d \leq 7$ dimensional maximally supersymmetric Yang-Mills theories on a class of curved backgrounds with the off-shell supercharges. The class 1 admits the nontrivial rigid supersymmetry if and only if the space dimension is four. In this case, the theory is conformally equivalent to the one on the flat space and we can actually recover the action by performing Weyl transformation from the flat action as in [15, 28]. The class 2 is more interesting. This class is admitted for the space dimension $d \leq 7$ and contains the various concrete examples: we have explicitly shown that S^d and AdS_d are involved in the class and the number of the off-shell supersymmetries is nine. By orbifolding S^3 , the lens space S^3/\mathbb{Z}_n preserves at least four off-shell supercharges. This class also involves Sasaki-Einstein and nearly Kähler manifolds for $d = 5$ and $d = 6$, respectively [49, 50, 51] although we did not concretely investigate these cases.

In our off-shell construction, supersymmetries are realized linearly. Thus it is transparent what types of the bosonic symmetries are generated by the commutator of the supersymmetric transformations. Especially we emphasize that the dilatation symmetry is not generated in the class 2: this is a necessary condition for existence of supersymmetries of non-conformal field theories at quantum level. In general, higher-dimensional field theories are not well-defined at quantum level. However when off-shell supercharges Q persist at quantum level, the localization formula make it possible to give a constructive formulation of such quantum theories at least in Q -closed sector. It is interesting to study quantum effects and work out localization procedures in these classes to challenge unsolved problems in superstring/M-theory [2].

Extension to other classes of backgrounds would be also interesting. For instance, we could twist by a line bundle L and consider the following Killing spinor equation as [38]

$$(\nabla_\mu - i\tilde{A}_\mu)\epsilon = -iV_\mu\epsilon - iV^\nu\Gamma_{\mu\nu}\epsilon, \quad (6.1)$$

where \tilde{A}_μ is a connection on L and V^μ is a smooth, conserved vector field. Furthermore, extension to backgrounds with torsion [54] and pseudo-Riemannian manifolds [55] would be also illuminating.

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A. Clifford algebra

Let γ^M be the standard gamma matrix in the 10d Minkowski space whose metric is given by $ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_9^2$. Then, γ^M satisfies the standard anti-commutation relation:

$$\{\gamma^M, \gamma^N\} = 2g^{MN}, \quad (A.1)$$

where $M, N = 1, \dots, 9, 0$ and γ^M is 32×32 matrix. Since we now consider even dimension, we have the chiral operator

$$\gamma^{11} = \gamma^1 \gamma^2 \dots \gamma^9 \gamma^0 \quad (\text{A.2})$$

and γ^M reverse chirality. Then, we can write γ^M as

$$\gamma^M = \begin{pmatrix} 0 & \tilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix}. \quad (\text{A.3})$$

Then, the “half” gamma matrices satisfies the following anti-commutation relation:

$$\tilde{\Gamma}^{\{M} \Gamma^{N\}} = g^{MN}, \quad \Gamma^{\{M} \tilde{\Gamma}^{N\}} = g^{MN}. \quad (\text{A.4})$$

And we choose Γ^M and $\tilde{\Gamma}^M$ to be symmetric.

Here we mention some useful formula.

- Triality identity:

$$(\Gamma_M)_{\alpha_1 \alpha_2} (\Gamma^M)_{\alpha_3 \alpha_4} = 0 \quad (\text{A.5})$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 1, \dots, 16$.

- Fierz identity for $d = 10$ Majorana-Weyl spinors with same chirality:

$$\begin{aligned} (\epsilon_1 A \epsilon_2)(\epsilon_3 B \epsilon_4) = \frac{1}{16} \Big[& (\epsilon_1 \epsilon_4)(\epsilon_3 B A \epsilon_2) - \frac{1}{2}(\epsilon_1 \tilde{\Gamma}_{MN} \epsilon_4)(\epsilon_3 B \tilde{\Gamma}^{MN} A \epsilon_2) \\ & + \frac{1}{4!}(\epsilon_1 \tilde{\Gamma}_{MNKL} \epsilon_4)(\epsilon_3 B \tilde{\Gamma}^{MNKL} A \epsilon_2) \Big] \end{aligned} \quad (\text{A.6})$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ are real spinors with 16 components, and A and B are 16×16 matrices.

- Other useful formula:

$$\begin{aligned} \tilde{\Gamma}^{NP} \Gamma^M &= \Gamma^{NPM} - 2g^{M[N} \Gamma^{P]}, \quad \tilde{\Gamma}_\rho \Gamma^{NP\rho} = \tilde{\Gamma}_\rho (\tilde{\Gamma}^{NP} \Gamma^\rho + 2g^{\rho[N} \Gamma^{P]}), \\ \tilde{\Gamma}_\rho \tilde{\Gamma}^{\mu\nu} \Gamma^\rho &= (d-4) \Gamma^{\mu\nu}, \quad \tilde{\Gamma}_\rho \tilde{\Gamma}^{\mu A} \Gamma^\rho = (d-2) \Gamma^{\mu A}, \quad \tilde{\Gamma}_\rho \tilde{\Gamma}^{AB} \Gamma^\rho = d \Gamma^{AB}. \end{aligned} \quad (\text{A.7})$$

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